Problem

Show that any point of the unit square $[0,1]^2$ is a rational Casino.

Solution

The problem is equivalent to showing that for any positive integer n a set $S_n \subset (0,1)^2 \cap \mathbb{Q}^2$ of rational coin pairs exists such that S_n is dense in the open unit square $(0,1)^2$ and the n-sided dice can be generated via a finite-time algorithm using any coin pair in S_n. From now on we thus simply fix n.

As a further notation, the coin whose heads probability equals $p \in (0,1)$ will be referred to in the following as the p-coin and the pair of coins with heads probabilities equal to $p \in (0,1)$ and $q \in (0,1)$ respectively will be called the (p,q)-coin pair.

The first fact we use is derived from the existence of the Lorenzo Strip: a set $T_n \subset (0,1/4) \cap \mathbb{Q}$ exists such that T_n is dense in $(0,1/4)$ and the n-sided dice can be generated using the pair of coins $(1/2,t)$ for any $t \in T_n$. Moreover, we can extend the set T_n to a dense set of rationals in $(0,1/2)$ by observing that if the $(1/2,t)$-coin pair generates the n-sided dice, then also the $(1/2,2t)$-coin pair does it: the HH event in the algorithm of tossing first the $1/2$-coin and then the $2t$-coin has probability equal to t. This shows that the $(1/2,2t)$-coin pair can be used to generate the t-coin too. Moreover, the set T_n can now be extended by symmetry to $(0,1)$ as using a p-coin is equivalent to using a $(1-p)$-coin. We thus have shown:

Fact 1. A set $T_n \subset (0,1) \cap \mathbb{Q}$ exists such that T_n is dense in $(0,1)$ and the n-sided dice can be generated using the pair of coins $(1/2,t)$ for any $t \in T_n$.

The second fact we need is that using an arbitrary (rational) coin we can generate a set of (rational) coins dense in (0,1), as follows.

Fact 2. Considering the t-coin for an arbitrary $t \in (0,1) \cap \mathbb{Q}$, a set $U_t \subset (0,1) \cap \mathbb{Q}$ exists that is dense in $(0,1)$ and such that for any $u \in U_t$ the u-coin can be generated via a finite-time algorithm using only the t-coin.

To see this, consider N an arbitrary positive integer and the algorithm of tossing the t-coin N times in a row, recording all 2^N possible outcomes $X_1, X_2, ..., X_{2^N}$ with their probabilities. Denoting by $z \in (0,1)$ the largest of t and $1-t$, all these 2^N probabilities are bounded from above by z^N. For a given $k = 1,2,...,2^N$ we now consider the final states of the algorithm to be $X_1 \cup X_2 \cup \cdots \cup X_k$ ('heads') and the complementary event $X_{k+1} \cup X_{k+2} \cup \cdots \cup X_{2^N}$ ('tails'). In this way we generate the u_k-coin where u_k is the cumulated probability of the first k outcomes $X_1, X_2, ..., X_k$. Obviously

$$0 = u_0 < u_1 < u_2 < \cdots < u_{2^N-1} < u_{2^N} = 1$$

and the difference between two consecutive u_k's equals the probability of an X outcome hence does not exceed z^N. Collecting all sequences $(u_k)_{1 \le k \le 2^N}$ for all positive integers N we obtain a dense set of rationals in $(0,1)$ since $z^N \to 0$ as $N \to \infty$. The proof of Fact 2 is complete.

Let us now combine Facts 1 and 2 to construct the set S_n. We simply take

$$S_n := \{(1/(2u), t) \mid t \in T_n \text{ and } u \in U_t \cap (1/2, 1)\}$$

where T_n and U_t are the notations in the statements of Facts 1 and 2 above.

The density of S_n in the unit square follows trivially from the density of T_n and U_t in $(0,1)$. It remains thus to show that the n-sided dice can be generated using any coin pair in S_n. To this end note that since $u \in U_t \cap (1/2, 1)$, by Fact 2 a finite-time algorithm A_u exists to generate the u-coin using only the t-coin. Consider now the coin pair $(1/(2u), t) \in S_n$ and the algorithm of tossing the u-coin once (i.e. running A_u), followed by one toss of the $(1/(2u)$-coin. The HH event of this algorithm has probability $u * (1/(2u)) = 1/2$ hence we have generated the fair coin. The fair coin and the t-coin are however sufficient to generate the n-sided dice since by assumption $t \in T_n$. The proof is thus complete. $\diamond$