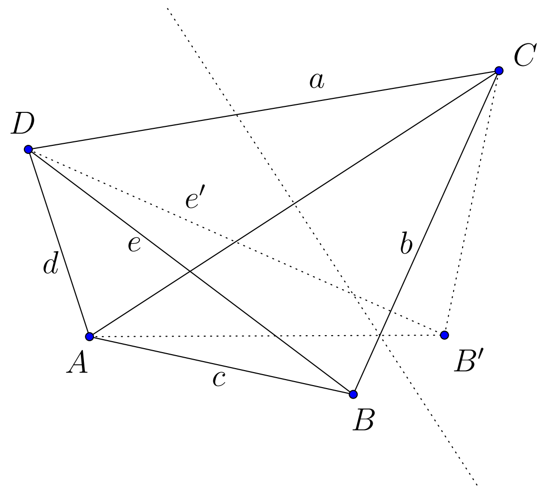
The present short mathematical note is devoted to the analysis of the inequality

$$e + f \leq L(a, b, c, d) := xa + yb + zc + td \tag{1}$$

with $(x, y, z, t) \in \mathbb{R}^4$ and that should hold for any quadrilateral $ABCD$ with side and diagonal lengths $a \geq b \geq c \geq d$ and e, f respectively. We obtain a complete, explicit characterization of the quadruples (x, y, z, t) fulfilling this condition.

Note 1

In proving a generic estimate of the type $e + f \leq L(a, b, c, d)$ that should hold for an arbitrary quadrilateral $ABCD$ with side and diagonal lengths $a \geq b \geq c \geq d$ and e, f respectively one can assume w.l.o.g. that $ABCD$ is convex and also that the lengths a and b correspond to opposite sides. To prove for example the latter claim assume the sides labeled a and b are adjacent, as shown in figure below. We deduce that B, D lie on the same side of the bisector line of the segment AC. Considering B' to be the reflection of B in this bisector line we have that $e = |DB| \leq |DB'| = e'$ whereas the quadrilaterals $ABCD$ and $AB'CD$ have the same four side lengths. It follows that if we are able to prove $e' + f \leq L(a, b, c, d)$ then $e + f \leq L(a, b, c, d)$ follows too.



It can be similarly shown that w.l.o.g. one can assume that $ABCD$ is a convex quadrilateral, i.e. A, C are on different sides of the line BD and also B, D are on different sides of the line AC. Supposing for example that A, C lie on the same side of the line BD, considering C' obtained by mirroring C in the line BD we deduce that $ABC''D$ is, in the sense of the inequality to be shown, a "worse" quadrilateral than $ABCD$ since $|AC'| \geq |AC|$ and $|BC| = |BC''|, |DC| = |DC''|$. Repeating this construction several times we reach in a finite number of steps a convex quadrilateral.

Proposition 1
Let $ABCD$ be a quadrilateral whose side and diagonal lengths equal $a \geq b \geq c \geq d$ and e, f respectively. Show that

$$(e+f)^2/4 \leq ab + cd + (a-b)^2/4 \tag{2}$$

and the equality is attained only by isosceles trapezoids with opposite parallel sides c, d and opposite equal sides $a = b$.

Similarly we have also

$$(e+f)^2/4 \leq ab + cd + (c-d)^2/4 \tag{3}$$

with equality attained only by isosceles trapezoids with opposite parallel sides a, b and opposite equal sides $c = d$.

Proof
Before presenting the proof let us briefly compare (2), (3) with the classical Ptolemy inequality known to hold for arbitrary quadrilaterals,

$$ef \leq ab + cd. \tag{4}$$

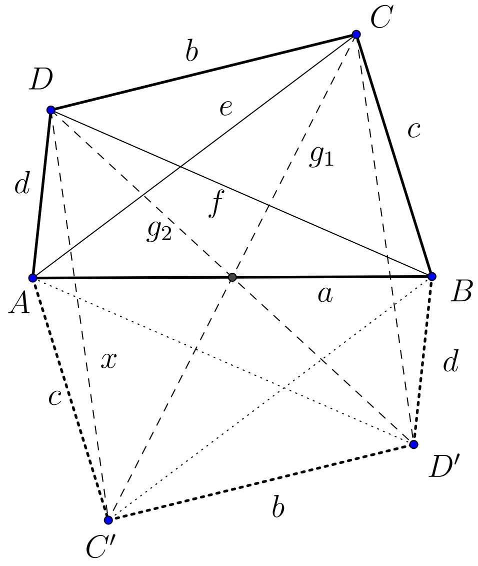
We clearly have $ef \leq (e+f)^2/4$ and examples can be easily found (e.g. numerically) to show that $(e+f)^2/4 \leq ab + cd$ does not always hold. The Ptolemy inequality (4) can thus not be directly improved upon by replacing the squared geometric mean by the squared arithmetic mean (of the two diagonal lengths) on the l.h.s. and without increasing the r.h.s. too. The estimates (2), (3) show us therefore two ways to amend also the r.h.s. of (4) and recover generic inequalities of Ptolemy type for the arithmetic mean of the two diagonal lengths. It is further interesting to note that (2), (3) become strictly stronger than the Ptolemy inequality for quadrilaterals whose largest (or smallest) two sides have the same length, i.e. $a = b$ or $c = d$.

We present below a complete proof of (2) and note already now that the same argument delivers symetrically also (3).

Returning now to the proof of (2) we first note that due to Note 1 above we can assume w.l.o.g. that $ABCD$ is a convex quadrilateral and a, b and c, d represent pairs of opposite sides. W.l.o.g. we can further assume the quadrilateral $ABCD$ to be labeled as represented in the Figure below, with the lengths of AB, BC, CD, DA, AC, BD equal to a, b, c, d, e, f respectively.

Let us now consider the reflections of the vertices C, D w.r.t. the midpoint of AB, which we denote by C' respectively D'. As shown in the Figure below, the lengths of CC' and DD' are denoted by g_1, g_2 whereas x represents the length of DC'.

Several parallelograms have been formed via this construction, most notably $CDC'D'$, $ADBD'$ and $ACBC'$. By a standard argument based on the cosine law we have



$$g_1^2 + g_2^2 = 2(b^2 + x^2)$$
$$a^2 + g_2^2 = 2(d^2 + f^2)$$
$$a^2 + g_1^2 = 2(c^2 + e^2)$$

from which by summing up all three equalities we immediately obtain

$$a^2 + b^2 + x^2 = c^2 + d^2 + e^2 + f^2.$$

The triangle inequality applied in $\Delta ADC'$ ensures $c + d \geq x$ so that

$$a^2 + b^2 + (c + d)^2 \geq c^2 + d^2 + e^2 + f^2$$

or, equivalently,

$$(e^2 + f^2)/2 \leq (a^2 + b^2)/2 + cd = ab + cd + (a - b)^2/2 \tag{5}$$

which is already an estimate of Ptolemy type, for the squared quadratic mean of the two diagonal lengths.

We briefly stop here to note that (5) seems to present some interest in itself too, as an inequality of Ptolemy type valid for arbitrary quadrilaterals and with equality attained iff the opposite sides denoted c, d are parallel.

The desired conclusion (2) follows now by interpolation between (5) and the classical Ptolemy inequality (4) with mixing weight 1/2, that is adding together half of (5) and half of (4).

Equality in (2) is equivalent to equalities in (5) and (4), that is $ABCD$ cyclic and possesing two parallel sides c, d which is in turn equivalent to $ABCD$ being an isosceles trapezoid with equal opposite sides $a = b$ and parallel sides c, d, or $ABCD$ being a degenerate quadrilateral with A, B, C, D all aligned and positioned in this order. The proof is complete. $\diamond$

Proposition 2
If $x, y, z, t \geq 0$ then the linear estimate

$$e + f \leq L(a, b, c, d) := xa + yb + zc + td \tag{1}$$

holds for any quadrilateral $ABCD$ with side and diagonal lengths equal to $a \geq b \geq c \geq d$ and e, f respectively if and only if (1) holds for the set S of quadrilaterals defined as $S := S_{\text{deg}} \cup S_{\text{isotr}}$ where

1. S_{deg} denotes the set of all degenerate quadrilaterals $ABCD$ with $a = b+c+d$,
2. S_{isotr} denotes the set of all isosceles trapezoids $ABCD$ with three equal side lengths, i.e. $a = b = c$ or $b = c = d$.

Proof
If (1) holds for any quadrilateral $ABCD$ then it trivially holds in particular also for all quadrilaterals belonging to S.
The converse however is not straigthforward and we now show that it follows in fact from the nonlinear estimate derived in Proposition 1 and that we can more conveniently reformulate as

$$e + f \leq \min\left(\sqrt{(a+b)^2 + 4cd}, \sqrt{(c+d)^2 + 4ab}\right). \tag{6}$$

From now on we thus assume that (1) holds for all quadrilaterals belonging to the set S and prove that it then holds for arbitrary quadrilaterals too. Moreover, we prove that the following stronger estimate holds for an arbitrary quadrilateral (and in the usual notations of side and diagonal lengths),

$$\min\left(\sqrt{(a+b)^2 + 4cd}, \sqrt{(c+d)^2 + 4ab}\right) \leq L(a, b, c, d). \tag{7}$$

At first sight it might seem odd that a stronger estimate such as (7) can be proved from the (a-posteriori) weaker inequality (1) applied to S, a subset of the set of all quadrilaterals. The fact that allows this to happen is that for the quadrilaterals belonging to S the estimate (6) holds *with equality.* This can be easily verified by applying the Ptolemy equality for the isosceles trapezoids (as cyclic quadrilaterals) and by a simple algebraic calculation for the degenerate quadrilaterals elements of S_{deg}, as the minimum in (6) is attained by the second square root term and equals $a + b$.

Returning now to the actual proof we wanted to present, we first note that obviously the minimum in (6), (7) is attained by the first square root term if and only if $a - b \leq c - d$, so that in the following we will distinguish two cases.

Additionally, a fact that we will repeatedly use in our analysis is the convexity of the function

$$(0,\infty) \ni x \mapsto \sqrt{x^2+\alpha}$$

for any $\alpha \geq 0$. In particular, if F represents a linear function in x and $I \subseteq (0,\infty)$ denotes an arbitrary interval then the inequality

$$\sqrt{x^2+\alpha} \leq F(x)$$

holds on I if and only if it holds on ∂I (i.e. boundary of I).

Case 1: $a-b \leq c-d$. To prove is the estimate

$$\sqrt{(a+b)^2+4cd} \leq L(a,b,c,d). \tag{8}$$

By $a-b \leq c-d$ we have that $a \in [b, b+c-d]$ so that in order for (8) to hold it is sufficient (by the above-mentioned convexity property) to have (8) fulfilled by the "boundary values" of a or, equivalently, by the tuples (b,b,c,d) and $(b+c-d,b,c,d)$, i.e.

$$\begin{aligned} 2\sqrt{b^2+cd} &\leq L(b,b,c,d) \\ \sqrt{(2b+c-d)^2+4cd} &\leq L(b+c-d,b,c,d). \end{aligned}$$

Now since $b \in [c,\infty)$, the two inequalities above hold if they are valid for $b \to \infty$ and $b = c$ (invoking again the convexity of the l.h.s.'s above as functions of b). The former (i.e. $b \to \infty$) is equivalent to (8) for the tuple $(1,1,0,0)$ whereas the latter (i.e. $b = c$) is equivalent to the same (8) for the tuples (c,c,c,d) and $(2c-d,c,c,d)$. Let us look at these three tuples in more detail.

Estimate (8) for the tuple $(1,1,0,0)$ is obviously equivalent to (1) for the degenerate quadrilateral with side lengths 1,1,0,0 and belonging therefore to S_{deg}. It thus holds by assumption.

Estimate (8) for the tuple (c,c,c,d) is obviously equivalent to (1) for the isosceles trapezoid with side lengths c,c,c,d and belonging therefore to S_{isotr}. It thus holds by assumption.

Estimate (8) for the tuple $(2c-d,c,c,d)$ is obviously equivalent to

$$\sqrt{(3c-d)^2+4cd} = \sqrt{(c-d)^2+8c^2} \leq L(2c-d,c,c,d)$$

and since $c-d \in [0,c]$ the same convexity argument shows that in order for this estimate to hold it is sufficient to have it fulfilled for $c-d=0$ and $c-d=c$. This is equivalent to (8) for the tuples (c,c,c,c) and $(2c,c,c,0)$ and these two conditions are nothing but (1) applied to a generic square, itself an element of S_{isotr}, respectively (1) for the degenerate quadrilateral with side lengths $2c,c,c,0$ and belonging thus to S_{deg}. These estimates hold then again by assumption and the proof of Case 1 is complete.

Case 2: $c-d \leq a-b$. To prove is

$$\sqrt{(c+d)^2+4ab} \leq L(a,b,c,d). \tag{9}$$

We employ arguments similar to those presented in the analysis of Case 1. We have that $d \in [c+b-a, c]$ and since $d \geq a-b-c$ (by the triangle inequality) we deduce $d \in [|c+b-a|, c]$. By the above-mentioned convexity property it is thus sufficient to have (9) hold for the tuples (a,b,c,c) and $(a,b,c,|c+b-a|)$ i.e.

$$\begin{array}{rcll} 2\sqrt{c^2+ab} & \leq & L(a,b,c,c) & \\ \sqrt{(2c+a-b)^2+4ab} & \leq & L(a,b,c,c+b-a) & \text{if} \quad c+b-a \geq 0 \\ \sqrt{(a-b)^2+4ab} & \leq & L(a,b,c,a-b-c) & \text{if} \quad c+b-a \leq 0 \end{array}$$

From $c \geq d \geq |a-b-c|$ we obtain $c \in [(a-b)/2, b]$ and invoking once more the convexity of the l.h.s.'s above as functions of c we deduce that it is sufficient to have (9) hold for the tuples (a,b,b,b), $(a,b,(a-b)/2,(a-b)/2)$, $(a,b,b,2b-a)$ for $2b-a \geq 0$, $(a,b,a-b,0)$ and $(a,b,c,a-b-c)$ for $a-b-c \geq 0$. Let us now discuss these tuples in more detail.

Estimate (9) for the tuple (a,b,b,b) is equivalent to (1) for the isosceles trapezoid with side lengths a,b,b,b and belonging to the class of quadrilaterals defined under point 2 of our problem formulation. It thus holds by assumption.

Estimate (9) for the tuples $(a,b,(a-b)/2,(a-b)/2)$, $(a,b,a-b,0)$ and $(a,b,c,a-b-c)$ for $a-b-c \geq 0$ which all represent degenerate quadrilaterals with $a = b+c+d$ therefore elements of S_{deg}. It is thus follows from (1) for S_{deg} which holds by assumption.

It remains to analyze (9) for the tuple $(a,b,b,2b-a)$ under the additional assumption $2b-a \geq 0$. This is equivalent to

$$\sqrt{(3b-a)^2+4ab} = \sqrt{(a-b)^2+8b^2} \leq L(a,b,b,2b-a)$$

and since $a-b \in [0,b]$ the same convexity argument ensures that this estimate is fulfilled if it holds for the tuples (b,b,b,b) and $(2b,b,b,0)$, representing a square, itself element of S_{isotr}, respectively a degenerate quadrilateral belonging to S_{deg}. In these two cases the desired estimate (9) is equivalent to (1) and holds thus by assumption. The proof of Case 2 and hence of Proposition 2 are now complete. $\diamond$

We now describe explicitly the set of all tuples $(x, y, z, t) \in \mathbb{R}^4$ such that the main linear estimate (1) is satisfied for any quadrilateral $ABCD$.

Proposition 3
If $(x, y, z, t) \in \mathbb{R}^4$ then the linear estimate

$$e + f \leq L(a, b, c, d) := xa + yb + zc + td \tag{1}$$

holds for any quadrilateral with side and diagonal lengths $a \geq b \geq c \geq d$ and e, f respectively if and only if

$$\begin{aligned}
x + y &\geq 2 && (10)\\
2x + y + z &\geq 3 && (11)\\
3x + y + z + t &\geq 4 && (12)\\
x + y + z + t &\geq 2\sqrt{2} \qquad \text{if } t \leq 1/\sqrt{2} \text{ or } x \geq 1/\sqrt{2} && (13)\\
x + y + z &\geq t + 1/t \qquad \text{if } 1/\sqrt{2} \leq t \leq 1 && (14)\\
x + y + z &\geq 2 \qquad \text{if } t \geq 1 && (15)\\
y + z + t &\geq x + 1/x \qquad \text{if } 1/2 \leq x \leq 1/\sqrt{2} && (16)
\end{aligned}$$

Proof
From Proposition 2 we know that (1) holds for arbitrary quadrilaterals if and only if it holds for all quadrilaterals elements of $S = S_{\text{deg}} \cup S_{\text{isotr}}$. We show that (1) for S_{deg} is equivalent to (10), (11), (12) whereas (1) for S_{isotr} delivers all other conditions.

Starting with S_{deg} we simply note that in a degenerate quadrilateral with $a = b + c + d$ we have $e + f = a + b = 2b + c + d$ hence (1) is equivalent to

$$(x + y - 2)b + (x + z - 1)c + (x + t - 1)d \geq 0 \quad \forall b \geq c \geq d \geq 0.$$

Due to linearity this is equivalent to the same estimate satisfied by the three tuples $(b, 0, 0), (b, b, 0), (b, b, b)$ and the resulting three conditions are exactly (10), (11), (12).

Turning now to S_{isotr} and denoting the sides of the isosceles trapezoid by (a, a, a, m) where $0 \leq m \leq 3a$ (here m denotes the length of the fourth side, that can be smaller or larger than a), the (double) condition to be satisfied reads

$$\begin{aligned}
2\sqrt{a^2 + am} &\leq (x + y + z)a + tm \quad \forall 0 \leq m \leq a && (17)\\
2\sqrt{a^2 + am} &\leq xm + (y + z + t)a \quad \forall a \leq m \leq 3a. && (18)
\end{aligned}$$

To analyze (17) we note that the function $[0, a] \ni m \mapsto 2\sqrt{a^2 + am} - tm$ has its first order derivative equal to $(1 + m/a)^{-1/2} - t$ hence a maximum at $m_0 = a$ if

$t \leq 1/\sqrt{2}$ or at $m_0 = (1/t^2 - 1)a$ if $1/\sqrt{2} \leq t \leq 1$, or at $m_0 = 0$ if $t \geq 1$. Therefore (17) holds iff it is satisfied at m_0, which is then equivalent to (13) (under the restriction on t only), (14), (15).

The analysis of (18) is similar. The function $[a, 3a] \ni m \mapsto 2\sqrt{a^2 + am} - xm$ has its first order derivative equal to $(1 + m/a)^{-1/2} - x$ hence a maximum at $m_0 = 3a$ if $x \leq 1/2$ or at $m_0 = (1/x^2 - 1)a$ if $1/2 \leq x \leq 1/\sqrt{2}$, or at $m_0 = a$ if $x \geq 1/\sqrt{2}$. Therefore (18) holds iff it is satisfied at these three m_0's (under the corresponding assumptions on x), resulting immediately in three conditions equivalent to (12), (13) (under the restriction on x only) and (16). $\diamond$