Using Your Head is Permitted
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> Can you tile a convex polygon with a finite number of concave quadrilaterals?
>
> To be considered a solver, either give a proof of impossibility or construct an example tiling.
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> For completeness, a shape is called convex if for any two points in it, the entire straight line segment between them is also in it. Concave is the opposite. And by *tiling* we mean a partitioning of the area of the polygon.

We will show that every polygon tiled in concave quadrilaterals is itself concave.

First we prove the following lemma.

Lemma 1. *Let P be a polygon tiled into n subpolygons, the set of whose we denote $\mathcal{C}$. Then there exists an order of $\mathcal{C}$, $\{C_i\}_{1\leq i\leq n}$ such that for every $k \leq n$*

$$\bigcup_{i=1}^{k} C_i$$

is simply connected.

Proof. Let C_1 be an arbitrary element of $\mathcal{C}$. Inductively we choose C_k. Assume $C_1, \ldots, C_k$ have been chosen such that $\cup_{i=1}^{k} C_i$ is simply connected. Let $\mathcal{C}_k = \mathcal{C} \setminus \{C_i\}_{i\leq k}$ be the set of remaining C_i.

Assume there is no $C_{k+1} \in \mathcal{C}_k$ such that $\bigcup_{i=1}^{k+1} C_i$ is simply connected. Let us choose the $C' \in \mathcal{C}$ such that $\bigcup_{i=1}^{k} \cup C'$ is connected, and the number of elements in $\mathcal{C}$ that are in the holes formed by the connected (but not simply connected) $\bigcup_{i=1}^{k} \cup C'$ is minimal under all choices of C'. Let us now choose C'' in one of the aforementioned holes, such that $\bigcup_{i=1}^{k} \cup C''$ is connected (and again it is not simply connected). The holes in $\bigcup_{i=1}^{k} \cup C''$ contain less elements of $\mathcal{C}$ than the holes in $\bigcup_{i=1}^{k} \cup C'$, hence we could not choose a C' satisfying the minimality condition. Thus there is always a C_{k+1} to choose such that $\bigcup_{i=1}^{k+1} C_i$ is simply connected.

(We define the *holes* to be the connection components of $\mathbb{R} \setminus \bigcup_{C \in \mathcal{C}_k} C$ that are finite.) ☐

Let us now assume a polygon P is tiled into n concave quadrilaterals. By Lemma 1 we can enumerate the quadrilaterals by $Q_1, \ldots, Q_n$, such that the union of the first k quadrilaterals $P_k = \bigcup_{i=1}^{k} Q_i$ is simply connected for all k, and hence a polygon.

For any polygon F we define the quantities $p(F), n(F), d(F)$ as follows:

- $p(F)$ is the number of convex corners of F, i.e. corners with inner angles less than π.
- $q(F)$ is the number of concave corners of F, i.e. corners with inner angles larger than π.
- $d(F) = p(F) - q(F)$.

Note that for a concave quadrilateral Q we have $p(Q) = 3, q(Q) = 1, d(Q) = 2$.

Lemma 2. *Given a polygon F and a concave quadrilateral Q, such that their interiors are disjoint and $F \cup Q$ is simply connected, we have $d(F \cup Q) \leq d(F)$.*

Proof. Let D be the intersection of Q and F. By the above criteria, D is either a point or a sequence of line segments.

In the case D is a point, we have one of the following cases

- D lies on a convex corner of F, and on the concave corner of Q. Instead of those two corners, $F \cup Q$ has two concave corners, hence

$$d(F \cup Q) = d(F) - 1 + d(Q) + 1 - 2 = d(F)$$

- D lies on a concave corner of F, and on a convex corner of Q. Instead of those two corners, $F \cup Q$ has two concave corners, hence

$$d(F \cup Q) = d(F) + 1 + d(Q) - 1 - 2 = d(F)$$

- D lies on convex corners of both F and Q. From the two generated corners of $F \cup Q$ at least one is concave, hence

$$d(F \cup Q) \leq d(F) - 1 + d(Q) - 1 - 0 = d(F)$$

- D lies on a convex corner of either F or Q, and on a boundary element of the other. Both new generated corners of $F \cup Q$ are concave, hence

$$d(F \cup Q) = d(F) + d(Q) - 1 - 2 = d(F) - 1$$

Let us now assume D is a connected sequence of line segments. We denote the two endpoints by A and B, as depicted in Figure 2

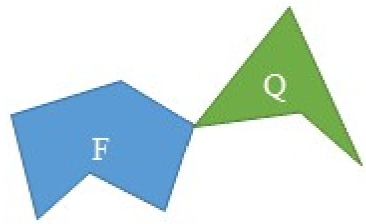

Figure 1: Example of when D is a point

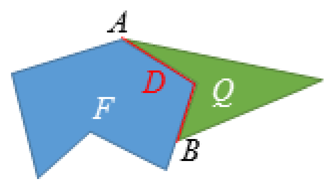

Figure 2: Example of when D is a line

Again we want to estimate $d(F \cup Q)$. We do this by observing the quantity $V = d(F \cup Q) - (d(F) + d(Q))$. We split V in three peices:

V_I counting all corners of F and Q that end up in the interior of $F \cup Q$. They don't provide to $d(F \cup Q)$, and each one provides $+1$ for one of $d(F)$ or $d(Q)$ and -1 for the other. Hence $V_I = 0$.

V_O counting all corners of F and Q that end up in the boundary of $F \cup Q$ and are not equal A or B. Each of them provides ± 1 to $d(F \cup Q)$ as well as to $d(F) + d(Q)$. Thus again, $V_O = 0$.

What is left is to consider V_{AB}, counting all corners of F, Q, and $F \cup Q$ lying in A or B.

For both A or B, the following cases can occur:

- Convex corner of F and convex corner of Q, convex/concave, or no corner of $F \cup Q$. In this case V is reduced by $1, 3$, or 2 respectively.
- Convex corner of F and concave corner of Q, and as a consequence a concave corner of $F \cup Q$. In this case V is reduced by 1. The same holds for concave corner of F and convex corner of Q.
- Convex corner of F , no corner (enterior of an edge) of Q, and as a consequence a concave corner of $F \cup Q$. In this case, V is reduced by 2.

We conclude that for each A and B, V_{AB} is reduced by at least 1, resulting in $V_{AB} \leq -2$. Thus

$$V = V_I + V_O + V_{AB} \leq -2$$

and consequentially

$$d(F \cup Q) = d(F) + d(Q) + V \leq d(F) + d(Q) - 2 = d(F).$$

□

We can now apply this lemma on P. We know $d(P_1) = d(Q_1) = 2$

As $P_{i+1} = P_i \cup Q_{i+1}$, we conclude from Lemma 2 that $d(P_{i+1}) \leq d(P_i)$. Inductively we get

$$d(P) = d(P_n) \leq d(P_1) = 2.$$

Any convex polygon R with nonempty interior (this is what we obviously are dealing with here) has $d(R) = p(R) \geq 3$. It can thus not be the union of finitely many concave quadrilaterals.